



Transformations, Partitions, and Endomorphisms

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Transformations, Partitions, and Endomorphisms

- Defining interesting monoids
- Diagram of embeddings
- Whats the relationship between M and $\text{End}(M)$?
- Sequences of endomorphism monoids (Endomorphism Towers):

$$\text{End}(M), \text{End}_2(M), \text{End}_3(M), \dots$$

Partial Bijections

Let X be a (finite) set, Y and Z be subsets of X , and $t : Y \rightarrow Z$ be a bijection.

- The **domain** of t is $\text{dom}(t) = \{x \in X : xt \text{ is defined}\}$,
- The image of t is $\text{im}(t) = \{y \in X : y = xt \text{ (for some } x \in X)\}$,
- The rank of t is the cardinality of $\text{im}(t)$.

The set of all **partial bijections** from $[1, n]$ into $[1, n]$ forms a monoid $\mathcal{I}_n$.

The Cayley Theorem for Finite Groups

All finite groups embed into the symmetric group S_n .

The Cayley(ish) Theorem for Finite Inverse Semigroups

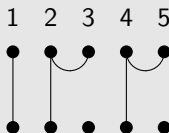
All finite **inverse semigroups** embed into $\mathcal{I}_n$.

Diagrams of Transformations

Let $t \in \mathcal{T}_n$ be the map

$$t = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 2 & 4 & 4 \end{pmatrix}.$$

We can illustrate t as a graph of $[1, n] \times [1, n]$



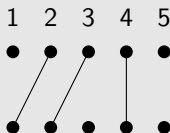
Diagrams made using the PMDraw package for Tikz by Matthias

Diagrams of Partial Bijections

Let $t \in \mathcal{I}_n$ be the map

$$t = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \emptyset & 1 & 2 & 4 & \emptyset \end{pmatrix}.$$

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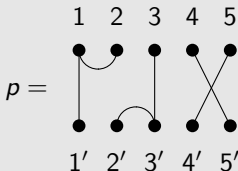


The Partition Monoid - $\mathcal{P}_n$

Let $X = [1, n]$ be a set and $X' = \{x' : x \in X\}$. Elements of $\mathcal{P}_n$ are partitions of $X \cup X'$. Example:

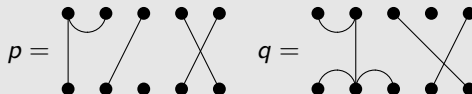
$$p = \{\{1, 2, 1'\}, \{3, 2', 3'\}, \{4, 5'\}, \{5, 4'\}\}$$

Illustrated as:

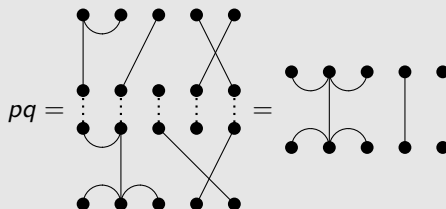


The Partition Monoid - $\mathcal{P}_n$

The product of two partitions in $\mathcal{P}_n$ is as follows. Define two partitions:

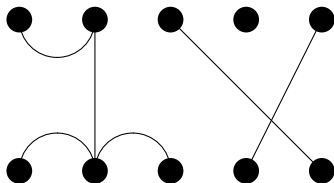


The product is concatenation of paths in the product diagram:



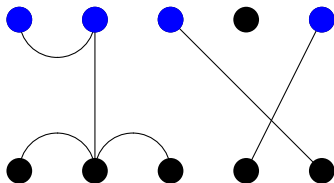
Aspects of a partition

- $\text{dom}^\wedge(p) = \{x \in X : x \text{ connects to some } y' \in X'\}$
- $\text{dom}_\vee(p) = \{x \in X : x' \text{ connects to some } y \in X\}$
- $\text{ker}^\wedge(p) = \{(x, y) \in X \times X : x \text{ and } y \text{ are connected}\}$
- $\text{ker}_\vee(p) = \{(x, y) \in X \times X : x' \text{ and } y' \text{ are connected}\}$
- $\text{rank}(p) = \text{the number of blocks in the partition that include elements of } X \text{ and } X'$



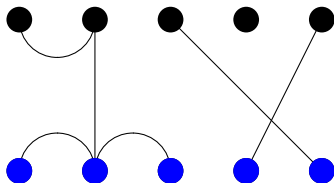
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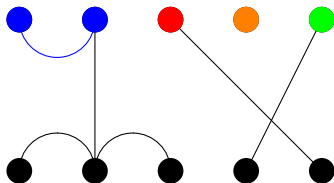
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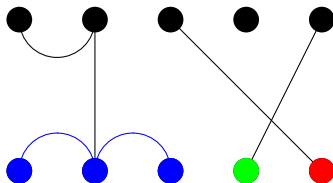
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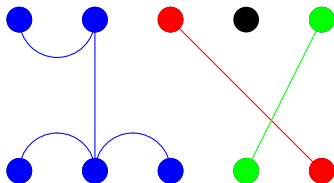
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- $\text{rank}(p)$ = the number of blocks in the partition that include elements of X and X'



Transformations as Partitions

All $t \in \mathcal{T}_n$ are partitions in $\mathcal{P}_n$ such that

- $\text{dom}^\wedge(t) = X$
- $\text{ker}_\vee(t) = \{(x, x) : x \in X\}$

Symmetries as Partitions

All $t \in \mathcal{S}_n$ are partitions in $\mathcal{P}_n$ such that

- $\text{dom}^\wedge(t) = \text{dom}_\vee(t) = X$
- $\text{ker}^\wedge(t) = \text{ker}_\vee(t) = \{(x, x) : x \in X\}$

Partial Bijections as Partitions

All $t \in \mathcal{I}_n$ are partitions in $\mathcal{P}_n$ such that

- $\text{ker}^\wedge(t) = \text{ker}_\vee(t) = \{(x, x) : x \in X\}$

Endomorphisms

Let S be a semigroup. An endomorphism of S is a map ϕ from S onto itself such that $x\phi y\phi = (xy)\phi$.

- i.e. a homomorphism from S to S .
- The set of endomorphisms forms a monoid $\text{End}(S)$.

Automorphisms

An automorphism of S is a bijective map ψ from S onto itself such that $x\psi y\psi = (xy)\psi$.

- The set of automorphisms forms a group $\text{Aut}(S)$.

Singular Endomorphism

An endomorphism that is not an automorphism is called a singular endomorphism.

- The set of singular endomorphisms forms a semigroup $\text{SEnd}(S)$

The Inner Automorphisms

If G is the group of units of M then the maps $\psi_g : x \rightarrow g^{-1}xg$ are *inner* automorphisms. $\text{Inn}(M) \cong G/C_M(G)$.

Inner Automorphisms of $\mathcal{S}_n, \mathcal{I}_n, \mathcal{T}_n, \mathcal{P}_n$

All these monoids have group of units $\mathcal{S}_n$.

- All our monoids have group of units $\mathcal{S}_n$ (which is centreless).
- $\text{Inn}(M) \cong \mathcal{S}_n$ for $M \in \{\mathcal{S}_n, \mathcal{I}_n, \mathcal{T}_n, \mathcal{P}_n\}$.

The Singular Endomorphisms

The singular endomorphism $\phi : S \rightarrow S$ consist of two parts:

- The kernel $\ker(\phi)$: is a congruence (or for groups it is a normal subgroup),
- The image $\text{im}(\phi)$: is a subsemigroup of S isomorphic to $S/\ker(\phi)$.

The Symmetric Group

Well known: $\text{Aut}(\mathcal{S}_n) = \{\psi_g : g \in \mathcal{S}_n\}$

For the singular endomorphisms, look at the potential kernel classes:

$$\{1\} \trianglelefteq \mathcal{A}_n \trianglelefteq \mathcal{S}_n$$

- Automorphisms: $\psi_g : x \rightarrow g^{-1}xg$ for all $g \in \mathcal{S}_n$
- Singular Endomorphisms:

$$\phi_g : x \rightarrow \begin{cases} g & \text{if } x \in \mathcal{S}_n \setminus \mathcal{A}_n \\ 1 & \text{if } x \in \mathcal{A}_n \end{cases}$$

where $g^2 = 1$.

Full Transformation Monoid (Schein and Teclezghi)

Possible kernel classes (congruences):

$$id \subseteq R_1 \subseteq R_{S_2} \subseteq R_2 \subseteq R_{\mathcal{A}_3} \subseteq \cdots \subseteq R_{n-1} \subseteq R_{\mathcal{A}_n} \subseteq R_{S_n} \subseteq \mathcal{T}_n \times \mathcal{T}_n$$

- Automorphisms: $\psi_g : x \rightarrow g^{-1}xg$ where $g \in S_n$
- Singular Endomorphisms:

$$\phi_{t,e} : x \rightarrow \begin{cases} t & \text{if } x \in S_n \setminus \mathcal{A}_n \\ t^2 & \text{if } x \in \mathcal{A}_n \\ e & \text{if } x \in \mathcal{T}_n \setminus S_n \end{cases}$$

where $t^3 = t$ and $te = et = e$.

Full Transformation Monoid (Schein and Teclezghi)

Possible kernel classes (congruences):

$$id \subseteq R_{\mathcal{A}_n} \subseteq R_{\mathcal{S}_n} \subseteq \mathcal{T}_n \times \mathcal{T}_n$$

- Automorphisms: $\psi_g : x \rightarrow g^{-1}xg$ where $g \in \mathcal{S}_n$
- Singular Endomorphisms:

$$\phi_{t,e} : x \rightarrow \begin{cases} t & \text{if } x \in \mathcal{S}_n \setminus \mathcal{A}_n \\ t^2 & \text{if } x \in \mathcal{A}_n \\ e & \text{if } x \in \mathcal{T}_n \setminus \mathcal{S}_n \end{cases}$$

where $t^3 = t$ and $te = et = e = e^2$.

The Partition Monoid (Mazorchuk)

Possible kernel classes (congruences):

$$id \subseteq \mathcal{I}_{n-1} \subseteq R_{\mathcal{A}_n} \subseteq R_{\mathcal{S}_n} \subseteq \mathcal{P}_n \times \mathcal{P}_n$$

- Automorphisms: $\psi_g : x \rightarrow g^{-1}xg$ where $g \in \mathcal{S}_n$
- Singular Endomorphism injective on the units:

$$\lambda_{g,e} : x \rightarrow \begin{cases} g^{-1}xg & \text{if } x \in \mathcal{S}_n \\ e & \text{otherwise} \end{cases}$$

where $ex = xe = e$ for all $x \in \mathcal{S}_n$.

- Singular Endomorphisms like those on $\mathcal{T}_n$:

$$\phi_{t,e} : x \rightarrow \begin{cases} t & \text{if } x \in \mathcal{S}_n \setminus \mathcal{A}_n \\ t^2 & \text{if } x \in \mathcal{A}_n \\ e & \text{if } x \in \mathcal{P}_n \setminus \mathcal{S}_n \end{cases}$$

where $t^3 = t$ and $te = et = e = e^2$.

The Symmetric Inverse Monoid (Schein and Teclezghi)

Types of endomorphism for $\mathcal{I}_n$:

- $\psi_g : x \rightarrow g^{-1}xg$ where $g \in \mathcal{S}_n$

- $\phi_{t,e} : x \rightarrow \begin{cases} t & \text{if } x \in \mathcal{S}_n \setminus \mathcal{A}_n \\ t^2 & \text{if } x \in \mathcal{A}_n \\ e & \text{if } x \in \mathcal{I}_n \setminus \mathcal{S}_n \end{cases}$

where $t^3 = t$ and $te = et = e = e^2$.

- $\lambda_{g,e} : x \rightarrow \begin{cases} g^{-1}xg & \text{if } x \in \mathcal{S}_n \\ e & \text{otherwise} \end{cases}$

where $ex = xe = e$ for all $x \in \mathcal{S}_n$.

- $\varepsilon_g : x \rightarrow \begin{cases} g^{-1}xg & \text{if } x \in \mathcal{S}_n \\ \hat{x} & \text{if } \text{rank}(x) = n - 1 \\ \emptyset & \text{otherwise.} \end{cases}$

where $g \in \mathcal{S}_n$ and $\hat{x}$ is the element $\text{dom}(\hat{x}) = X \setminus \text{dom}(x)$, image is $\text{im}(\hat{x}) = X \setminus \text{im}(x)$.

Exceptional Endomorphisms of $\mathcal{I}_n$ and $\mathcal{P}_n$

The monoids $\mathcal{P}_n$ and $\mathcal{I}_n$ have the additional endomorphisms

$$\lambda_{g,e} : x \rightarrow \begin{cases} g^{-1}xg & \text{if } x \in \mathcal{S}_n \\ e & \text{otherwise,} \end{cases}$$

$g \in \mathcal{S}_n$, $e^2 = e$, $\forall h \in \mathcal{S}_n$ $he = eh = e$.

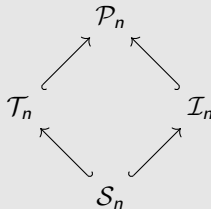
The monoid $\mathcal{I}_n$ has the additional endomorphisms

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$g \in \mathcal{S}_n$, $\hat{x}$ is the idempotent whose image contains the only element not in the image of x , $\emptyset$ is the empty map.

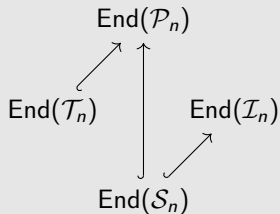
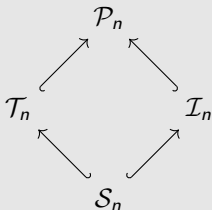
Embeddings of M and $\text{End}(M)$

Our monoids of partitions and transformations have the following diagram of inclusion:



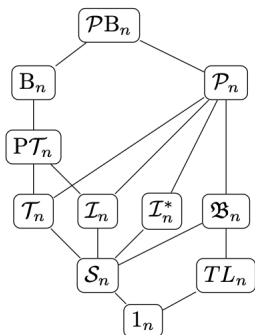
Embeddings of M and $\text{End}(M)$

Our monoids of partitions and transformations have the following diagram of inclusion:



Question

Is there some M such that $N \hookrightarrow M$ and $\text{End}(N) \hookrightarrow \text{End}(M)$ for $N \in \{\mathcal{S}_n, \mathcal{I}_n, \mathcal{T}_n, \mathcal{P}_n\}$?



Symbol	Name and References
$\mathcal{PB}_n$	Partitioned binary relations [25]
$\mathcal{B}_n$	Binary relations [27]
$\mathcal{PT}_n$	Partial transformation semigroup [15]
$\mathcal{P}_n$	(Bi)partition monoid [17]
$\mathfrak{B}_n$	Brauer monoid
$\mathcal{S}_n$	Symmetric group [4, 9]
$\mathcal{T}_n$	Full transformation semigroup [15]
$\mathcal{I}_n$	Symmetric inverse monoid [22]
$\mathcal{I}_n^*$	Dual symmetric inverse monoid [13]
TL_n	Temperley-Lieb, Jones monoid J_n

Diagram from: Finite Diagram Semigroups: Extending the Computational Horizon.
By: East, Ergi-Nagy, Francis, Mitchell

Endomorphism Towers and Embeddings

Let M be a monoid and $\text{End}(M)$ its endomorphism monoid. Define a sequence,

$$\text{End}(M), \text{End}_2(M) = \text{End}(\text{End}(M)), \text{End}_3(M) = \text{End}(\text{End}(\text{End}(M))), \dots$$

What can we say about the relationship between the endomorphism monoids in this sequence?

- For all $M \in \{\mathcal{S}_n, \mathcal{T}_n, \mathcal{I}_n, \mathcal{P}_n\}$, $\text{End}(M)$ is described.
- What do we know about the structure of $\text{End}(\text{End}(M))$?

$$\text{End}(M) \hookrightarrow \text{End}(\text{End}(M)).$$

- Does $\text{End}_i(M) \hookrightarrow \text{End}_{i+1}(M)$ for all i ?
- What of the embeddings between $\text{End}(\text{End}(M))$ for our various monoids?

So what next?

- Q. Is there some natural choice N such that $\text{End}(M) \hookrightarrow \text{End}(N)$ for $\mathcal{S}_n$, $\mathcal{T}_n$, etc?
- Q. Do our embeddings for $\text{End}(M)$ extend to $\text{End}_2(M)$?
- Q. Can we determine an embedding pattern for sequences of endomorphisms?

$$\text{End}(M) \hookrightarrow \text{End}_2(M) \hookrightarrow \text{End}_3(M) \hookrightarrow \dots?$$